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Quantencomputing und Quantensimulation
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Aufgabe 22: Landau-Zener formula (5 points)

Consider the Hamiltonian operator of a two-level system (qubit),

$$H(t) = \begin{pmatrix} \frac{-\alpha t}{2} & V \\ V & \frac{\alpha t}{2} \end{pmatrix}.$$

a) (oral) Sketch the time dependence of the eigenvalues of the Hamiltonian, for $V = 0$ and for an arbitrary $V > 0$.

b) (1 point) Assume that the wavefunction is given by $|\psi(t)\rangle = (c_1(t), c_2(t))^T$. Show that the coefficients of the wavefunction for $V = 0$ are $c_{1/2}(t) = c_{1/2}(0) \exp\left(\pm \frac{i\alpha t^2}{4\hbar}\right)$.

c) (2 points) Let us consider the case $V > 0$. In order to obtain an approximation, we use variation of parameters on the solution of problem b),

$$c_{1/2}(t) = \tilde{c}_{1/2}(t) \exp\left(\pm \frac{i\alpha t^2}{4\hbar}\right).$$

Show that the parameters $\tilde{c}_{1/2}(t)$ satisfy the system of differential equations,

$$i\hbar \frac{\partial \tilde{c}_1(t)}{\partial t} = \tilde{c}_2(t) V \exp\left(-\frac{i\alpha t^2}{2\hbar}\right),$$
$$i\hbar \frac{\partial \tilde{c}_2(t)}{\partial t} = \tilde{c}_1(t) V \exp\left(\frac{i\alpha t^2}{2\hbar}\right).$$

d) (2 points) Assume now that the system initially was in the lower energy level: $c_2(t \to -\infty) = 1$. Use the approximation $\tilde{c}_2(t) \approx 1$ to obtain $\tilde{c}_1(t \to \infty)$. Calculate the first order correction to $|c_2(t \to \infty)|^2$. What does $|c_2(t \to \infty)|^2$ describe? Compare this solution with the exact one,

$$|c_2(t \to \infty)|^2 = \exp\left(-\frac{2\pi V^2}{\hbar\alpha}\right).$$

Pro tip: Use $\int_{-\infty}^{\infty} \mathrm{d}x \exp(-ax^2) = \sqrt{\pi/a}$.

Aufgabe 23: Trotter-Suzuki Formula (5 points)

Given the Hamiltonian $H = A + B$ with $[A, B] \neq 0$, we want to calculate the corresponding time evolution operator $U(t) = \exp\left(-\frac{i}{\hbar}Ht\right)$ with the help of the Trotter-Suzuki formula.

a) (1 point) Compare the given time evolution operator with the following approximated form: $U_{TS}(t) = \exp\left(-\frac{i}{\hbar}At\right)\exp\left(-\frac{i}{\hbar}Bt\right)$. Show that these operators are identical up until second order in time $\mathrm{O}(t^2)$, therefore $U_{TS}(t)$ can be regarded as a linear approximation to $U(t)$.

b) (oral) What advantage does $U_{TS}(t)$ have over the more direct linear approximation $U(t) \approx \mathbb{1} - \frac{i}{\hbar}Ht$? *Hint: What properties should a time evolution operator fulfill?*

c) (2 points) Show that $U(t)$ can be approximated by $U(t) \approx U_{TS}(t/n)^n$ for arbitrarily long times t.

Now consider the rotation of a qubit along the axis $\boldsymbol{n} = \frac{1}{\sqrt{2}}(1, 0, 1)^T$ with an angle of α, given by $U(\alpha) = \exp\left(-i\alpha\boldsymbol{n}\cdot\boldsymbol{\sigma}/2\right)$. We now want to use the Trotter Suzuki formula to represent this rotation by alternating rotations around the $\hat{\boldsymbol{x}}$- and $\hat{\boldsymbol{z}}$-axis.
d) (2 points) Define $U_{TS}(\alpha)$. Sketch $\left|\langle+|\, U^\dagger(\alpha)U_{TS}(\alpha/n)^n\, |+\rangle\right|^2$ for three different rotation angles α as a function of n and discuss the results.
Note: 1 point for three different rotation angles with $n = 1$ and one point for the dependence on n. The dependence on n can be calculated using MATLAB or MATHEMATICA.